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Dolan–Grady relations and noncommutative quasi-exactly solvable systems

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Abstract
We investigate a $U(1)$ gauge invariant quantum mechanical system on a 2D noncommutative space with coordinates generating a generalized deformed oscillator algebra. The Hamiltonian is taken as a quadratic form in gauge covariant derivatives obeying the nonlinear Dolan–Grady relations. This restricts the structure function of the deformed oscillator algebra to a quadratic polynomial. The cases when the coordinates form the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ algebras are investigated in detail. Reducing the Hamiltonian to 1D finite-difference quasi-exactly solvable operators, we demonstrate partial algebraization of the spectrum of the corresponding systems on the fuzzy sphere and noncommutative hyperbolic plane. A completely covariant method based on the notion of intrinsic algebra is proposed to deal with the spectral problem of such systems.

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1. Introduction

The Dolan–Grady relations (DGR) arose 20 years ago in a very interesting paper [1], where Dolan and Grady considered a class of systems described by a Hamiltonian given as a linear combination of the two operators obeying some nonlinear relations. The relations, afterwards called by their names, guarantee the existence of an infinite set of mutually commuting charges which includes the Hamiltonian. This naturally connects the construction to the integrable systems [1–5]. A rich algebraic structure associated with the Dolan–Grady relations was revealed by Perk [2] and Davies [3] who showed that the two operators satisfying DGR generate recursively the infinite-dimensional Onsager algebra. The latter appeared in a seminal work [6] on exact solution of the planar Ising model. The set of commuting charges constructed by Dolan and Grady is linear in generators of the Onsager algebra.

Recently, it was observed [7–9] that the Dolan–Grady relations arise as necessary conditions for the anomaly-free quantization of pseudo-classical systems with nonlinear holomorphic supersymmetry (n-HSUSY). In accordance with the general algebraic construction of the n-HSUSY [9], the Hamiltonian depending on a natural parameter n is a quadratic form in two mutually conjugate operators subjected to DGR, whereas the supercharges are (anti)holomorphic polynomials of order n in the same operators. It was shown that the n-HSUSY Hamiltonian belongs to an infinite set of mutually commuting even operators quadratic in the Onsager algebra generators, and that these even integrals together with the odd supercharges form a polynomial superalgebra of the order n. Realization of this construction in various physical systems [8, 10, 11] revealed an intimate relation of the n-HSUSY with the quasi-exactly solvable (QES) systems [12].

Though the quadratic set of commuting integrals admits any real and even complex values for the parameter n playing the role of a coupling constant, such an extension breaks the nonlinear supersymmetry. Nevertheless, in [11] it was observed that the quasi-exact solvability of the systems associated with the quadratic set of integrals of motion roots not in the n-HSUSY construction itself, but rather in the Dolan–Grady relations. In particular, it was shown that the corresponding systems can be quasi-exactly solvable not only for integer values of the coupling constant n.

In the present paper, exploiting the Dolan–Grady relations we consider a class of $U(1)$ gauge invariant quantum mechanical systems on a two-dimensional (2D) noncommutative manifold, and demonstrate the quasi-exact solvability of their spectral problem for any value of the coupling constant.

The layout of the paper is as follows. In section 2 the basic information on the Dolan–Grady relations and Onsager algebra necessary for further analysis is presented in the context of integrable systems, n-HSUSY and low-dimensional matrix models. Realization of the Dolan–Grady relations in terms of the generalized deformed oscillator algebra is given in section 3. In section 4 we construct a $U(1)$ gauge invariant quantum mechanical system with noncommutative coordinates generating the generalized deformed oscillator algebra. In section 5 we demonstrate that by a reduction to finite-difference equations, the spectral problem of the model on the fuzzy sphere and noncommutative hyperbolic plane can be partially solved for some values of the coupling constant. In section 6 we develop an algebraic covariant approach to deal with the spectral problem, and show that really our noncommutative system is quasi-exactly solvable for any value of the coupling constant. The obtained results are summarized in section 7. In the appendix, we discuss a deformation of the $\mathfrak{sl}(2,\mathbb{R})$ scheme associated with the finite-difference quasi-exactly solvable equations.

2. Dolan–Grady relations

2.1. Dolan–Grady relations and Onsager algebra

The Dolan–Grady relations [1] can be represented in the form [9, 10]

$$[Z,\ [Z,\ [Z,\ \bar{Z}]]] = \omega^2[Z,\ \bar{Z}] \qquad [\bar{Z},\ [\bar{Z},\ [Z,\ \bar{Z}]]] = \bar{\omega}^2[Z,\ \bar{Z}]. \tag{2.1}$$

Here Z and $\bar{Z}$ are some operators which we shall call generating elements while ω and $\bar{\omega}$ are some constants. The nonlinear relations (2.1) can be used to construct the infinite set of mutually commuting operators for some integrable systems. In the most elegant way they can be represented in terms of the Onsager algebra [6] spanned by the generators A_n and G_n having the commutation relations

$$[A_m,\ A_n] = 4G_{m-n} \qquad [G_m,\ A_n] = 2A_{n+m} - 2A_{n-m} \qquad [G_m,\ G_n] = 0 \tag{2.2}$$

where $m, n \in \mathbb{Z}$. In fact, the operators $A_0 \sim \bar{Z}$ and $A_1 \sim Z$ are the generating elements of the Onsager algebra obeying the Dolan–Grady relations and all the other operators are defined recursively by relations (2.2) [3]. The values of the constants ω and $\bar{\omega}$ can be fixed by rescaling the generators. Then, the infinite set of the mutually commuting operators constructed by Dolan and Grady [1] is represented as

$$2J_m = A_m + A_{-m} + \lambda(A_{m+1} + A_{1-m}) \tag{2.3}$$

where $\lambda \in \mathbb{R}$. The commutativity of the operators J_m follows directly from the algebra (2.2). Now, treating the operator $J_0 = A_0 + \lambda A_1$ as a Hamiltonian, one obtains, generally, an integrable system with infinite number of conserved commuting charges. The charges of the form (2.3) appear in various models, such as the Baxter eight-vertex [5], the two-dimensional Ising [6] and Z_N spin models [4]. The power of the algebraic formulation (2.2), (2.3) is that it does not refer to the number of dimensions or to the nature of the spacetime manifold which can be lattice, continuum or loop space.

2.2. *Structure of n-HSUSY*

In the systems associated with the linear set of integrals (2.3) the operators Z ($\sim A_1$) and $\bar{Z}$ ($\sim A_0$) are supposed to be Hermitian. In [9] it was shown that the set (2.3) is not a unique set of mutually commuting operators which can be constructed in terms of generators of the Onsager algebra. There exists a set of commuting operators which are quadratic in the generators of the algebra (6.2). Such commuting operators naturally appear in the scheme of the n-HSUSY [8–11], which we discuss shortly below. Note, that for the systems associated with the quadratic set of commuting operators, the Z, $\bar{Z}$ and the ω, $\bar{\omega}$ are usually supposed to be mutually conjugate operators and complex constants, respectively.

The Hamiltonian of the system associated with the quadratic set of commuting charges (see equation (6.2) for their explicit form) reads

$$H_\lambda = \frac{1}{4}\{\bar{Z}, Z\} - \frac{\lambda}{4}[Z, \bar{Z}] \tag{2.4}$$

where a real parameter λ serves as a coupling constant, while mutually conjugate operators Z and $\bar{Z}$ are supposed to obey the nonlinear Dolan–Grady relations (2.1)[3].

When the coupling constant is integer, $\lambda = \pm n, n \in \mathbb{Z}_+$, the system (2.4) can be extended to the supersymmetric system with the Hamiltonian $\mathcal{H}_n = H_n\sigma_-\sigma_+ + H_{-n}\sigma_+\sigma_-$, $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$. In this case the Dolan–Grady relations (2.1) guarantee the existence of the odd integrals of motion (supercharges)

$$Q = \sigma_+ \prod_{k=0}^{n-1}\left(Z - \frac{\omega}{2}(2k - n + 1)\right) \qquad \bar{Q} = \sigma_- \prod_{k=0}^{n-1}\left(\bar{Z} - \frac{\bar{\omega}}{2}(2k - n + 1)\right). \tag{2.5}$$

The supercharges generate a nonlinear superalgebra of the order n. Its structure and further details of the algebraic construction underlying the nonlinear holomorphic supersymmetry can be found in [9].

The advantage of the supersymmetric formulation is that the zero modes of the supercharge Q (or $\bar{Q}$) form an invariant subspace with respect to the action of the Hamiltonian H_n (or H_{-n}). Furthermore, as a rule the zero modes are easier to calculate than the eigenfunctions of the Hamiltonian due to the factorized structure of the supercharges. When the number of zero

[3] In principle, one can also deal with Hermitian operators Z and $\bar{Z}$. Then the Hamiltonian (and the whole set of integrals (6.2)) is Hermitian if the coupling constant is pure imaginary [9]. However, here we will not discuss such systems.

modes is finite, then the Hamiltonian is a quasi-exactly solvable operator. The information on the functional form of the zero modes is very helpful for solving the corresponding spectral problem.

Although the system (2.4) formally admits an infinite number of integrals, nevertheless, for any given realization of the generating elements in a system with finite number of degrees of freedom this infinite set is reduced to a finite set of independent integrals. Such systems not being integrable in the general case, however, reveal a profound connection with quasi-exactly solvable systems [9]. We shall return to this aspect below.

2.3. Matrix models

Here we show that the Dolan–Grady relations can be revealed in some low-dimensional matrix models.

So, let us consider a matrix model given by the action

$$S = -\mathrm{Tr}\left(\tfrac{1}{2}[X_\mu, X_\nu][X^\mu, X^\nu] + R_{\mu\nu}X^\mu X^\nu\right) \tag{2.6}$$

where X_μ are Hermitian operators on a Hilbert space and $R_{\mu\nu}$ are c-numbers. The last term can be treated as describing the interaction of the vector field X_μ with an external symmetric field $R_{\mu\nu}$. One can also think that its diagonal part contains the mass term of the vector field. The matrix models of the form (2.6) are widely used in the context of string theories. In the case of a positively definite metric the action (2.6) corresponds to the potential part of the BFSS model [13] and defines static (vacuum) configurations of the theory. On the other hand, for the Lorentz metric the action can be interpreted as an effective theory of the reduced model of Yang–Mills theory [14] with the specific interaction. Actually, action (2.6) represents a slightly modified massive matrix model [15].

The action (2.6) leads to the equations of motion

$$[X^\nu, [X_\mu, X_\nu]] + R_{\mu\nu}X^\nu = 0. \tag{2.7}$$

In the two-dimensional case equation (2.7) can be represented in the form

$$\begin{aligned} &[X_-, [X_-, X_+]] + R_{+-}X_- + R_{--}X_+ = 0 \\ &[X_+, [X_+, X_-]] + R_{+-}X_+ + R_{++}X_- = 0 \end{aligned} \tag{2.8}$$

where $X_\pm = \frac{1}{\sqrt{2}}(X_1 \pm X_2)$ or $X_\pm = \frac{1}{\sqrt{2}}(X_1 \pm \mathrm{i}X_2)$ for the Lorentz or Euclidean metric, respectively. In the first case R_{++} and R_{--} are real constants, while in the latter $R^*_{--} = R_{++}$. It is easy to see that any solution of equation (2.8) also obeys the Dolan–Grady relations (2.1) with the identification:

$$X_- \sim Z \qquad X_+ \sim \bar{Z} \qquad R_{--} \sim -\omega^2 \qquad R_{++} \sim -\bar{\omega}^2.$$

Moreover, note that in the case of positively definite metric and when $R_{++} = R_{--} = 0$, equations (2.8) coincide with the basic commutation relations for the single-mode parafermion [16].

Now let us consider the following modification of the three-dimensional(3D) matrix model [17]

$$S = -\mathrm{Tr}\left(\tfrac{1}{2}[X_j, X_k]^2 - \tfrac{4}{3}\mathrm{i}\alpha\epsilon_{jkl}X_jX_kX_l + R_{jk}X_jX_k\right) \tag{2.9}$$

where α and R_{jk} are real numbers and we imply a positively definite metric. This model can be treated as the matrix model corresponding to the usual Yang–Mills theory with Chern–Simon term. As before, the quadratic term in action (2.9) represents a specific interaction with an external symmetric field. The equations of motion are

$$[[X_k, X_j], X_k] - \mathrm{i}\alpha\epsilon_{jkl}[X_k, X_l] + R_{jk}X_k = 0. \tag{2.10}$$

Imposing the condition

$$[X_0, X_\pm] = \pm\beta X_\pm \tag{2.11}$$

with $\beta \in \mathbb{R}$ on the components X_0, $X_\pm$, the equation for $j = 0$ is satisfied identically for $\beta = \alpha$, $R_{0i} = 0$. The equations of motion for $X_\pm$ have the form (2.8) with the substitution $R_{+-} \to R_{+-} + \alpha$. Therefore, the Dolan–Grady relations arise in this model as well. Again, like in model (2.6), for $R_{\pm\pm} = 0$ the equations of motion for the components $X_\pm$ take the form of basic commutation relations of the single-mode parafermion.

3. Generalized deformed oscillator

3.1. Main definitions

Another basic ingredient we shall use in what follows is the generalized deformed oscillator (GDO). Therefore, in this subsection we present a brief description of the construction (see also [18, 19]).

The generalized deformed oscillator is defined by the algebra generated by the set of operators $\{\mathbb{I}, z, \bar{z}, N\}$ satisfying the relations

$$[N, z] = -\theta z \qquad [N, \bar{z}] = \theta\bar{z} \qquad \bar{z}z = F(N) \qquad z\bar{z} = F(N+\theta) \tag{3.1}$$

where N is the number operator and the structure function $F(.)$ is an analytic function with the properties $F(0) = 0$ and $F(n\theta) > 0$, $n = 1, \ldots$. The structure function $F(.)$ is characteristic of the deformation scheme. Here we have introduced a positive real parameter θ, which will serve later on as a measure of noncommutativity.

The operators $z, \bar{z}$ obey the commutation relation

$$[z, \bar{z}] = \theta\Delta_+ F(N). \tag{3.2}$$

By symbols $\Delta_\pm$ we denote the forward and backward difference derivatives

$$\Delta_+ f(N) = \frac{f(N+\theta) - f(N)}{\theta} \qquad \Delta_- f(N) = \frac{f(N) - f(N-\theta)}{\theta}. \tag{3.3}$$

The generalized deformed oscillator algebra can be naturally represented on the Fock space of the eigenstates of the number operator N, $N|n\rangle = n\theta|n\rangle$, $\langle n|m\rangle = \delta_{nm}$, $n = 1, \ldots$, with the vacuum state defined as $z|0\rangle = 0$. The eigenstates are given by

$$|n\rangle = \frac{1}{\sqrt{F(n\theta)!}}\bar{z}^n|0\rangle \qquad F(n\theta)! \equiv \prod_{k=1}^{n} F(k\theta). \tag{3.4}$$

The operators z and $\bar{z}$ are the annihilation and creation operators of the oscillator algebra,

$$z|n\rangle = \sqrt{F(n\theta)}|n-1\rangle \qquad \bar{z}|n\rangle = \sqrt{F((n+1)\theta)}|n+1\rangle. \tag{3.5}$$

In what follows we shall denote the Fock space defined by (3.4), (3.5) as $\mathcal{H}_F$. On this space the operators z and $\bar{z}$ are mutually adjoint, $\bar{z} = z^\dagger$.

Note that when the characteristic function obeys the condition

$$F((p+1)\theta) = 0 \tag{3.6}$$

for some $p \in \mathbb{N}$, the creation–annihilation operators satisfy the nilpotency relations $z^{p+1} = \bar{z}^{p+1} = 0$. This means that the corresponding representation is $(p+1)$ dimensional and we have a parafermionic-type system of the order p.

3.2. The Dolan–Grady relations in terms of GDO

The simplest realization of the Dolan–Grady relations (2.1) can be obtained in terms of the generalized deformed oscillator:

$$Z = z \qquad \bar{Z} = \bar{z}. \tag{3.7}$$

As a consequence of the natural grading of the GDO algebra defined by the operator N, for this representation of the generating elements Z and $\bar{Z}$ it is possible to realize the Dolan–Grady relations only in its contracted form, $\omega = 0$ [9],

$$[Z,\ [Z,\ [Z,\ \bar{Z}]]] = 0 \qquad [\bar{Z},\ [\bar{Z},\ [Z,\ \bar{Z}]]] = 0. \tag{3.8}$$

On the other hand, these relations lead to the restriction on the structure function in the form of the finite-difference equation:

$$\Delta_+^3 F(N) = 0. \tag{3.9}$$

Since the structure function has to vanish at zero, the general solution of equation (3.9) is

$$F(N) = c_1 N^2 + c_0 \theta N \tag{3.10}$$

where *ad interim* $c_0, c_1 \in \mathbb{R}$. Three different cases corresponding to the solution (3.10) should be distinguished:

(a) The case $c_1 = 0, c_0 \neq 0$ corresponds to the usual oscillator given by the Heisenberg algebra. The nonlinear supersymmetry for such a system was discussed in [7].

(b) For $c_1 > 0$ the polynomial (3.10) is a structure function when the parameters obey the inequality $c_1 + c_0 > 0$. Then the generalized oscillator realizes the half-bounded infinite-dimensional discrete series of the unitary representations of the $\mathfrak{sl}(2, \mathbb{R})$ algebra. The generators are, correspondingly, $L_0 = \theta \Delta_+ F(N), L_- = z, L_+ = \bar{z}$, or $L_0 = -\theta \Delta_+ F(N), L_- = \bar{z}, L_+ = z$ for the series $\mathcal{D}_\alpha^+$ or $\mathcal{D}_\alpha^-$ [20] with $\alpha = (c_0 + c_1)/(2c_1)$.

(c) In the case $c_1 < 0$, for the solution (3.10) to be a structure function one has to impose the additional condition (3.6). This leads to quantization of one of the parameters, $c_0 = -(p+1)c_1$, and results in the $(p+1)$-dimensional representation of the $\mathfrak{su}(2)$ algebra with the generators $L_- = z, L_+ = \bar{z}$ and $L_3 = \theta \Delta_+ F(N)$. This is the single-mode parafermion case.

4. Noncommutative quantum mechanics and DGR

Let us consider a 2D quantum mechanical system with noncommutative complex-like coordinates $z, \bar{z}$ obeying the GDO algebra (3.1) realized on the Hilbert space $\mathcal{H}_F$. The operator-valued functions $\Psi(\bar{z}, z)$ on $\mathcal{H}_F$ with a finite norm with respect to the scalar product

$$(\Phi, \Psi) \equiv \mathrm{Tr}_{\mathcal{H}_F}(\Phi^\dagger(\bar{z}, z)\Psi(\bar{z}, z)) \tag{4.1}$$

will be treated by us as the states of the noncommutative system. Here the symbols $\mathrm{Tr}_{\mathcal{H}_F}$ and $\dagger$ denote the usual trace and Hermitian conjugation operations on the space $\mathcal{H}_F$ (3.4). The operator-valued functions with the finite norm themselves form a Hilbert space, which we shall denote as $\hat{\mathcal{H}}_F$.

The corresponding derivative operators on the space $\hat{\mathcal{H}}_F$ can be defined as

$$\partial \equiv -\theta^{-1}\,\mathrm{ad}\,\bar{z} \qquad \bar{\partial} \equiv \theta^{-1}\,\mathrm{ad}\,z. \tag{4.2}$$

These operators possess the usual conjugation property $\partial^\dagger = -\bar{\partial}$ with respect to the scalar product (4.1). It is worth noting that the derivative operators do not commute if the structure function $F(N)$ is different from a linear one. Indeed, $[\partial, \bar{\partial}] = \theta^{-1}\,\mathrm{ad}\,\Delta_+ F(N)$.

Let us introduce a $U(1)$ gauge interaction associated with an external magnetic field. The noncommutative system admits the following two types of gauge transformations,

$$\Psi'(\bar{z}, z) = U(\bar{z}, z)\Psi(\bar{z}, z) \qquad \Psi'^{\dagger}(\bar{z}, z) = \Psi^{\dagger}(\bar{z}, z)U^{\dagger}(\bar{z}, z) \tag{4.3}$$

or

$$\Psi'(\bar{z}, z) = U(\bar{z}, z)\Psi(\bar{z}, z)U^{\dagger}(\bar{z}, z) \tag{4.4}$$

with $U(\bar{z}, z)$ being a unitary operator. The scalar product (4.1) is invariant with respect to the both transformations. In the commutative limit, $\theta \to 0$, the transformation (4.3) is reduced to the usual $U(1)$ gauge transformation while (4.4) becomes trivial.

We realize the generating elements from (2.4) as the gauge covariant derivatives

$$Z = \partial + \rho[A(\bar{z}, z)] \qquad \bar{Z} = -\bar{\partial} + \rho[\bar{A}(\bar{z}, z)] \tag{4.5}$$

where the operator $\rho[\mathcal{O}]$, $\mathcal{O} \in \{A(\bar{z}, z), \bar{A}(\bar{z}, z)\}$, acts on Ψ, $\Psi^{\dagger}$ as

$$\rho[\mathcal{O}]\Psi = \mathcal{O}\Psi \qquad \rho[\mathcal{O}]\Psi^{\dagger} = -\Psi^{\dagger}\mathcal{O}$$

manifesting the fact that the states Ψ and $\Psi^{\dagger}$ carry the charges of the opposite sign. The operator-valued functions A and $\bar{A} = A^{\dagger}$ are the components of the noncommutative vector gauge potential corresponding to the case of stationary magnetic field, and we assume that the gauge coupling constant ('electric charge') and the mass parameter are equal to unity, $e = m = 1$. Under both gauge transformations (4.3) and (4.4) the components of the noncommutative vector potential are transformed in the usual way

$$A' = UAU^{\dagger} + U\partial U^{\dagger} \qquad \bar{A}' = U\bar{A}U^{\dagger} - U\bar{\partial}U^{\dagger}.$$

The commutator of the generating elements has the form

$$\mathcal{B} \equiv [Z, \bar{Z}] = -\theta^{-1}\,\mathrm{ad}\,\Delta_+ F(N) + B(\bar{z}, z). \tag{4.6}$$

We will treat

$$B(\bar{z}, z) \equiv \partial\bar{A} + \bar{\partial}A + [A, \bar{A}] \tag{4.7}$$

as an external quasi-magnetic field since in the commutative limit the commutator in (4.7) disappears and it turns into a true magnetic field. Nevertheless, in the noncommutative case the quantity (4.7) transforms as a connection under the gauge transformations. Actually, if one considers the operator N as the third dependent coordinate, then the quantity (4.7) is the Nth component of the gauge connection.

Let us suppose that the magnetic field depends on N only. This corresponds to an axially symmetric field in the commutative limit. The following choice of the gauge,

$$A(\bar{z}, z) = \bar{z}f(N) \qquad \bar{A}(\bar{z}, z) = f(N)z \tag{4.8}$$

where $f(.)$ is a real function, guarantees such a dependence of the magnetic field. Indeed, in the gauge (4.8) the quasi-magnetic field (4.7) acquires the form

$$B(N) = \Delta_+(F(N)f(N-\theta)(2-\theta f(N-\theta))). \tag{4.9}$$

Now let us analyse the restrictions which follow from the Dolan–Grady relations. In the representation (4.5) the lhs of equation (2.1) can be rewritten as follows:

$$\begin{aligned}\text{lhs (2.1)} = {} & -\theta^{-1}\,\mathrm{ad}\left(\bar{z}^2\Delta_+^3 F(N)\right) + \bar{z}^2\,(1-\theta f(N+\theta))\,(1-\theta f(N))\,\Delta_+^2 B(N) \\ & + \bar{z}^2(f(N) + f(N+\theta)(1-\theta f(N)))\Delta_+^3 F(N).\end{aligned} \tag{4.10}$$

The comparison of the first terms in (4.6) and (4.10) leads to the conclusion that the Dolan–Grady relations (2.1) with $\omega \neq 0$ can be satisfied only when the structure function of the GDO algebra is linear. On the other hand, for $\omega = 0$ expression (4.10) vanishes only when

the structure function obeys equation (3.9). This leads to the following constraint on the quasi-magnetic field:

$$\Delta_+^2 B(N) = 0. \tag{4.11}$$

In other words, the quasi-magnetic field has to be at most linear in N. The function $f(N)$ appeared in the gauge (4.8) can be found from (4.9) taking into account the constraint (4.11).

The classification of the GDO algebra with the quadratic structure function (3.10) was presented in the previous section. In accordance with it, the noncommutative coordinates $z, \bar{z}$ and the operator N form the $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, or the Heisenberg algebras. Therefore, the noncommutative system given by the Hamiltonian (2.4) in the representation (4.5) can be treated, correspondingly, as the system on the fuzzy sphere S_θ^2, on the noncommutative hyperbolic plane H_θ^2, or on the noncommutative plane $\mathbb{R}_\theta^2$.

Note that since the structure function is quadratic, the simplest solution for $f(N)$ is

$$f(N) = f = \text{const}. \tag{4.12}$$

Indeed, one can verify that in this case the quasi-magnetic field (4.7) is a linear function,

$$B(N) = f(2 - \theta f)(2c_1 N + (c_0 + c_1)\theta) \tag{4.13}$$

and hence, it obeys equation (4.11). Below we shall consider only the solution (4.12), (4.13). Later on we shall show that for the given solution expression (4.13) coincides with a covariant definition of the magnetic field in the gauge (4.8).

From the explicit form of the magnetic field (4.13) it follows that the parameter f has two special values $f = 0$ and $f = 2\theta^{-1}$. The first case is rather trivial. It corresponds to the model without gauge interaction (a free model (2.4) on the corresponding noncommutative space). In the second case the magnetic field (4.13) vanishes. Nevertheless, the gauge interaction still is present in the model since the corresponding connection (4.8) with $f \neq 0$ is not trivial. This resembles the famous Aharonov–Bohm effect but its origin roots here in the noncommutativity of the configuration space. As we shall see, there is also another critical value of the parameter, $f = \theta^{-1}$. For this value the quasi-magnetic field is nontrivial while the system is degenerate. Such a case, in turn, resembles the situation with existence of the critical value of the constant external magnetic field being coherent with the value of the parameter of noncommutativity (see, e.g., [21, 22]).

In the next section we shall discuss the spectral problem for the noncommutative system (2.4) on the fuzzy sphere and noncommutative hyperbolic plane.

5. Reduction to finite-difference QES systems

Here we show that the noncommutative system given by the Hamiltonian (2.4) is quasi-exactly solvable for some values of the coupling constant. It is demonstrated by a reduction of the Hamiltonian to one-dimensional finite-difference QES operators.

First of all, one can note that in the gauge (4.8) the operator $\operatorname{ad} N$ commutes with the Hamiltonian. One can refer this property to the existence in the system of an 'axial' symmetry. Therefore, the eigenstates of the Hamiltonian can be represented in the form

$$\Psi_L(\bar{z}, z) = \bar{z}^m \psi(N) \quad \text{or} \quad \Psi_R(\bar{z}, z) = \psi(N) z^m \tag{5.1}$$

where $m \in \mathbb{Z}_+$. Evidently, these operator functions are eigenstates of the operator $\operatorname{ad} N$. We shall call the functions Ψ_L and Ψ_R the left and right modes.

Now it is clear enough that the corresponding 2D spectral equation with the eigenstates of the form (5.1) can be reduced to a one-dimensional problem. But unlike the commutative

case, here the corresponding 1D system is represented by a finite-difference equation. One notes that such a situation is typical for any noncommutative system with 'axial' symmetry.

The Hamiltonian (2.4) is associated with the nonlinear holomorphic supersymmetry for integer values of the coupling constant. The zero modes of the supercharge Q form an invariant subspace of the Hamiltonian. We will use their functional form as an ansatz for investigating the spectral problem with arbitrary coupling constant.

Let us temporarily set $\lambda = n \in \mathbb{N}$. For the left mode wavefunctions represented in the factorized form

$$\Psi_L(\bar{z}, z) = \bar{z}^m \varphi(N)\phi(N)$$

with $\phi(N)$ obeying the equation

$$(\Delta_+ + f)\phi(N) = 0 \tag{5.2}$$

the zero mode equation $Z^n \Psi_L = 0$ is reduced to

$$(1 - \theta f)^n \Delta_+^n \varphi(N) = 0.$$

For $\theta f \neq 1$ this equation gives the polynomial solution $\varphi(N) = P_{n-1}(N)$, where $P_k(.)$ denotes a polynomial of the kth degree. For $\theta f = 1$ there are no zero modes at all, i.e. there are no bounded (normalized) states in the system.

The existence of the zero modes means that the Hamiltonian is quasi-exactly solvable when the coupling constant λ is a non-negative integer. Now we investigate the question on existence of the zero modes for an arbitrary value of the coupling constant. For the left modes the Hamiltonian can be reduced, $H_\lambda \Psi_L(\bar{z}, z) \to H^L_{\lambda,m}\psi(N)$, to the one-dimensional finite-difference operator

$$\begin{aligned} H^L_{\lambda,m} = {} & (\theta f - 1)(F(N)\Delta_+\Delta_- + (m+1)\mathcal{F}(N)\Delta_+) + f^2 F(N) \\ & + \tfrac{1}{2} f((2m + \lambda + 1)\theta f - 2(m + \lambda))\Delta_+ F(N) + \text{const} \end{aligned} \tag{5.3}$$

with the function $\mathcal{F}(N)$ defined by

$$\mathcal{F}(N) = \Delta_+ F(N) + \frac{m\theta}{2}\Delta_+^2 F(N). \tag{5.4}$$

For $\theta f = 1$ the reduced Hamiltonian (5.3) is a multiplicative operator. The generating elements of the form (4.5) are multiplicative operators as well, $Z\Psi \sim \Psi\bar{z}$, $\bar{Z}\Psi \sim \Psi z$. Actually, this representation of the Dolan–Grady relations becomes equivalent to the representation (3.7). However, the Hamiltonian (2.4) acting on the Hilbert space $\hat{\mathcal{H}}_F$ has a trivial dynamics in this case. Therefore, later on we shall always assume $\theta f \neq 1$.

The leading term of the 'potential' in (5.3) is proportional to the structure function $F(N)$. We represent the function $\psi(N)$ in the factorized form

$$\psi(N) = \varphi(N)\phi(N) \tag{5.5}$$

where the function $\phi(N)$ is supposed to obey the equation

$$(\Delta_+ + \gamma)\phi(N) = 0 \tag{5.6}$$

with a real parameter γ. This equation is a slight modification of equation (5.2). The factorization (5.5) allows us to discard the leading term in the equation for $\varphi(N)$ when the constant γ obeys the algebraic equation

$$\frac{\gamma^2}{1 - \theta\gamma} = \frac{f^2}{1 - \theta f} \tag{5.7}$$

that gives a possibility to look for the solutions $\varphi(N)$ in a polynomial form.

The first evident solution of equation (5.7)

$$\gamma = f \tag{5.8}$$

corresponds to the zero modes found above. Indeed, the reduction

$$H^L_{\lambda,m}\psi(N) \to H'_{\lambda,m}\varphi(N)$$

results in the one-dimensional finite-difference operator

$$H'_{\lambda,m} = -F(N)\Delta_+\Delta_- - (m+1)(1-\theta f)^2\mathcal{F}(N)\Delta_+ + f(2-\theta f)\mathcal{T}_\lambda + \text{const.} \tag{5.9}$$

Here, the function $\mathcal{F}(N)$ is defined by (5.4), while the operator $\mathcal{T}_\lambda$ is given by

$$\mathcal{T}_\lambda = F(N)\Delta_+ - \lambda\Delta_+F(N). \tag{5.10}$$

Let $\mathcal{P}_k(x), k \in \mathbb{Z}_+$, denote the space of all the polynomials in x up to the order k. On such spaces the finite-difference derivatives act similar to the usual derivative, $\Delta_\pm : \mathcal{P}_k(N) \to \mathcal{P}_{k-1}(N)$. Using this fact and that the function $F(N)$ is quadratic while $\mathcal{F}(N)$ is linear, one can note that the spaces $\mathcal{P}_k(N)$ are invariant subspaces of the operator (5.9) except for the third term. On the other hand, the space $\mathcal{P}_n(N)$ is an invariant subspace of the operator $\mathcal{T}_\lambda$ for any $\lambda = n \in \mathbb{Z}_+$. As a result, for non-negative integer values of the coupling constant the operator (5.9) is quasi-exactly solvable [12].

The second solution of equation (5.7) is

$$\gamma = \frac{f}{\theta f - 1}. \tag{5.11}$$

As a consequence, the reduced Hamiltonian acting on $\varphi(N)$ reads

$$H''_{\lambda,m} = -(1-\theta f)^2(F(N)\Delta_+\Delta_- + (m+1)\mathcal{F}(N)\Delta_+) - f(2-\theta f)\mathcal{T}_{-\lambda-2m} + \text{const.} \tag{5.12}$$

Therefore, this operator is quasi-exactly solvable if one imposes the condition $\lambda + 2m \in \mathbb{Z}_-$ on the coupling constant.

Now we pass over to the discussion of the right modes. Unfortunately, in this case we failed to find the zero modes of the supercharge Q in a closed form. Therefore, we shall investigate the conjectural partial algebraization of this part of the spectrum from the viewpoint of the reduction of the Hamiltonian.

For the right modes (5.1) the 2D Hamiltonian is reduced to the finite-difference operator

$$\begin{aligned} H^R_{\lambda,m} = {} & (\theta f - 1)(F(N)\Delta_+\Delta_- + (m+1)\mathcal{F}(N)\Delta_+) + f^2F(N) \\ & + \tfrac{1}{2}f((\lambda+1)\theta f + 2(m-\lambda))\Delta_+F(N) + \text{const.} \end{aligned} \tag{5.13}$$

This Hamiltonian can be obtained from (5.3) by means of the formal change $\lambda \to \lambda - 2m$. Using this property, one can reproduce all the corresponding results for the right modes. The factorization (5.5) with conditions (5.6) and (5.7) for the solution (5.8) leads to the finite-difference operator containing the term proportional to the operator $\mathcal{T}_{\lambda-2m}$. Hence, for $\lambda - 2m \in \mathbb{Z}_+$ the polynomial space $\mathcal{P}_{\lambda-2m}(N)$ is an invariant subspace of the obtained operator, i.e. it is a QES operator. For the second solution (5.11) the reduced Hamiltonian contains $\mathcal{T}_{-\lambda}$, and so it is a QES operator for non-positive integer values of the coupling constant.

All the reduced finite-difference operators, (5.9), (5.12) and those for the right modes, can be represented as QES operators of the form (A.10) with the following coefficients:

(1) The solution (5.8):

$$a_1 = 0 \quad a_2 = 0 \quad b^+_+ = -\frac{c_1}{\theta}(1-\theta f)^2 \quad b^+_0 = c_0 \quad b^+_- = 0 \quad b^-_+ = \frac{c_1}{\theta}$$

$$b^-_0 = -(c_0 + 2(m+1)c_1)(1-\theta f)^2 \quad b^-_- = -\theta(m+1)\,(c_0 + (m+1)c_1)\,(1-\theta f)^2$$

where $\mathsf{N} = \lambda$ or $\mathsf{N} = \lambda - 2m$ for the left or right modes, respectively.

(2) The solution (5.11):

$$a_1 = 0 \quad a_2 = 0 \quad b_+^+ = -\frac{c_1}{\theta} \quad b_0^+ = c_0(1-\theta f)^2 \quad b_-^+ = 0$$

$$b_+^- = \frac{c_1}{\theta}(1-\theta f)^2 \quad b_0^- = -c_0 - 2(m+1)c_1 \quad b_-^- = -\theta(m+1)\,(c_0 + (m+1)c_1)$$

where $\mathsf{N} = -\lambda - 2m$ or $\mathsf{N} = -\lambda$ for the left or right modes, respectively.

It is worth noting that for $\theta f = 2$ both operators (5.9) and (5.12) are exactly solvable since the term proportional to $\mathcal{T}_\lambda$ vanishes.

Let us briefly discuss a normalizability condition for the found solutions. Since the function $\varphi(N)$ is a polynomial, the normalizability has to be provided by the factor $\phi(N)$. The solution of equation (5.6) has the form $\phi(N) \sim (1-\theta\gamma)^N$. Therefore, the norm of the corresponding state $\Psi \in \hat{\mathcal{H}}_F$ can be represented as

$$\|\Psi\|_{\hat{\mathcal{H}}_F} = \mathrm{Tr}_{\mathcal{H}_F}(\Psi^\dagger\Psi) = C_{\mathcal{N}} \sum_{n=1}^{\infty} n^K (1-\theta\gamma)^{2n} + \cdots. \tag{5.14}$$

Here the dots denote less divergent terms in the sense that if the first term in (5.14) is convergent, then the others are convergent as well. The factor $C_{\mathcal{N}}$ is related to the normalization constant and the integer parameter K is defined by the type of solution. The requirement of convergence of the series leads to the condition

$$|1-\theta\gamma| < 1. \tag{5.15}$$

Therefore, we have to impose the following restrictions on the values of the parameter f: $\theta f \in (0, 2)$ for solution (5.8) and $\theta f \in (-\infty, 0) \cup (2, \infty)$ for solution (5.11).

Of course, in the case of the fuzzy sphere the norm (5.14) is finite because of the finiteness of the corresponding Hilbert space, and no additional condition appears for it.

6. Intrinsic algebra and quasi-exact solvability

In this section, we show that the noncommutative system under consideration really is quasi-exactly solvable for any value of the coupling constant. The demonstration is realized in terms of an intrinsic algebra.

6.1. Intrinsic algebra

In general, the operators $Z_0 \equiv Z$ and $\bar{Z}_0 \equiv \bar{Z}$ together with the contracted Dolan–Grady relations (3.8) recursively generate the infinite-dimensional contracted Onsager algebra [9]:

$$\begin{aligned} &[Z_k,\ \bar{Z}_l] = B_{k+l+1} \qquad [Z_k,\ B_l] = Z_{k+l} \qquad [B_k,\ \bar{Z}_l] = \bar{Z}_{k+l} \\ &[Z_k,\ Z_l] = 0 \qquad [\bar{Z}_k,\ \bar{Z}_l] = 0 \qquad [B_k,\ B_l] = 0 \end{aligned} \tag{6.1}$$

where $k, l \in \mathbb{Z}_+$ and $B_0 \equiv 0$ is implied. This algebra can be extended discarding the last condition, i.e. by treating B_0 as a nontrivial generator. Then the extended algebra will be generated by the set Z, $\bar{Z}$ and B_0. The element B_0 splits up the contracted Onsager algebra into the three subalgebras of grades $+1, 0$ and -1, given, correspondingly, by the span of $\bar{Z}_m$, B_m and Z_m. In other words, the generator B_0 is a grading element which provides the extended algebra with the triangular decomposition. This grading is different from that considered in [9].

The algebra (6.1) admits the infinite set of the commuting quadratic charges

$$\mathsf{J}_\lambda^k = \frac{1}{2}\sum_{p=1}^{k}\{\bar{Z}_{p-1}, Z_{k-p}\} - \frac{1}{2}\sum_{p=1}^{k-1} B_p B_{k-p} - \frac{\lambda}{2} B_k \tag{6.2}$$

with $k \in \mathbb{Z}_+$, which contains the Hamiltonian (2.4), $\mathsf{J}^1_\lambda = 2H_\lambda$, and the grading operator, $\mathsf{J}^0_\lambda \sim B_0$.

In the gauge (4.8) the generating elements of the form (4.5) read

$$Z = -\theta^{-1}\,\mathrm{ad}\,\bar{z} + f\rho[\bar{z}] \qquad \bar{Z} = -\theta^{-1}\,\mathrm{ad}\,z + f\rho[z].$$

These operators generate the following algebra intrinsic to the noncommutative system,

$$\begin{aligned}
&[Z,\ \bar{Z}] = 2c_1(S + f(2-\theta f)G) && [\mathcal{D},\ \bar{\mathcal{D}}] = -2c_1\theta G\\
&[Z,\ \bar{\mathcal{D}}] = 2c_1(1-\theta f)G && [G,\ \mathcal{D}] = \theta\mathcal{D}\\
&[\mathcal{D},\ \bar{Z}] = 2c_1(1-\theta f)G && [\bar{\mathcal{D}},\ G] = \theta\bar{\mathcal{D}}\\
&[Z,\ G] = (1-\theta f)\mathcal{D} && [S,\ \bar{\mathcal{D}}] = \bar{\mathcal{D}}\\
&[G,\ \bar{Z}] = (1-\theta f)\bar{\mathcal{D}} && [\mathcal{D},\ S] = \mathcal{D}\\
&[Z,\ S] = Z && [S,\ \bar{Z}] = \bar{Z}\\
&[S,\ G] = [Z,\ \mathcal{D}] = [\bar{Z},\ \bar{\mathcal{D}}] = 0
\end{aligned} \tag{6.3}$$

where the operators $\bar{\mathcal{D}}$, $\mathcal{D}$, G and S in the chosen gauge are

$$\mathcal{D} = \bar{z} \qquad \bar{\mathcal{D}} = z \qquad G = N + \theta\frac{c_0 + c_1}{2c_1} \qquad S = -\theta^{-1}\,\mathrm{ad}\,N. \tag{6.4}$$

The algebra has the following two Casimir operators:

$$\begin{aligned}
C_1 &= \{\bar{\mathcal{D}}, \mathcal{D}\} - 2c_1G^2\\
C_2 &= \{\theta\bar{Z} + (1-\theta f)\bar{\mathcal{D}}, \theta Z + (1-\theta f)\mathcal{D}\} - 2c_1(\theta S + G)^2.
\end{aligned} \tag{6.5}$$

The algebra (6.3) is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ ($c_1 > 0$) or $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ ($c_1 < 0$). In the chosen gauge it can be shown by representing the generators of the algebra in the form L_z, $L_{\bar{z}}$, L_N and R_z, $R_{\bar{z}}$, R_N, where L_a and R_a denote the left and right multiplications. In the given representation the Casimir operators (6.5) are equal to the same number,

$$C_1 = C_2 = \frac{\theta^2}{2c_1}\left(c_1^2 - c_0^2\right). \tag{6.6}$$

It is necessary to note that the values $\theta f = 0, 1, 2$ are special from the viewpoint of the algebra (6.3). Although the algebraic content of it remains the same, the algebra is not generated by the operators Z and $\bar{Z}$, i.e. it is not intrinsic any more. In these cases the generating elements produce the $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$ only.

The operators $\mathcal{D}, \bar{\mathcal{D}}$ and G can be treated as dependent covariant coordinates of a noncommutative membrane. Indeed, the equivalent set of coordinates

$$X_1 = \frac{\mathcal{D} + \bar{\mathcal{D}}}{\sqrt{8|c_1|}} \qquad X_2 = \mathrm{i}\frac{\mathcal{D} - \bar{\mathcal{D}}}{\sqrt{8|c_1|}} \qquad X_3 = G$$

obeys the condition

$$X_1^2 + X_2^2 - \varepsilon X_3^2 = \varepsilon\theta^2\alpha(1-\alpha) = -\varepsilon r^2 \tag{6.7}$$

where $\varepsilon = \mathrm{sign}\,c_1$ and $\alpha = (c_1 + c_0)/(2c_1)$ ($c_1 > 0$) or $\alpha = -p/2$ ($c_1 < 0$), $p \in \mathbb{Z}_+$. So, the operators X_i form the set of noncommutative covariant coordinates of the fuzzy sphere S^2_θ ($c_1 < 0$), or of the noncommutative hyperbolic plane H^2_θ ($c_1 > 0$). The operators Z, $\bar{Z}$ and S, in their turn, can be interpreted as the covariant derivatives. Note that in the noncommutative system under consideration, unlike the constant magnetic field case [22], the covariant coordinates are not proportional to the covariant derivatives.

The intrinsic algebra gives us the possibility to define the magnetic field in a covariant way. In general, if the derivatives form a Lie algebra, $[\partial_i, \partial_j] = f_{ij}{}^k\partial_k$, then the corresponding gauge covariant field strength can be defined as $\mathrm{i}F_{ij} = [\mathcal{D}_i, \mathcal{D}_j] - f_{ij}{}^k\mathcal{D}_k$ (e.g., see [22]),

where $\mathcal{D}_i$ are gauge covariant derivatives. From this definition and the intrinsic algebra it follows that in the present case only the 'transversal' component ($i = 3$) of the magnetic field, $B_i = \frac{1}{2} f_i{}^{jk} F_{jk}$, does not vanish. In the gauge (4.8) this component exactly coincides with the definition (4.13). It is interesting to note that as in the commutative case [11], the magnetic field is a linear function of the 'transversal' coordinate.

In terms of the generators of the algebra (6.3) the higher operators of the contracted Onsager algebra are

$$Z_m = (2c_1)^m (Z + \theta^{-1}(1-\theta f)(1-(1-\theta f)^{2m})\mathcal{D})$$
$$B_m = (2c_1)^m (S + \theta^{-1}(1-(1-\theta f)^{2m})G)$$

where $\theta f \neq 0, 1, 2$. Applying this form of the generators of the Onsager algebra to the infinite set of the commuting charges (6.2), one can obtain

$$\mathsf{J}^k_\lambda = \frac{(2c_1)^{k-1}}{\theta f(2-\theta f)}(2(1-(1-\theta f)^{2k})H_\lambda - c_1((1-\theta f)^2 - (1-\theta f)^{2k})S(S-\lambda)) + \cdots$$

where dots denote a linear combination of the Casimir operators (6.5). This representation leads to the conclusion that the Hamiltonian (2.4) and the operator S form a complete set of nontrivial integrals of the system under consideration.

6.2. *The commutative limit*

Our aim is to apply the found intrinsic algebra of the noncommutative system with the Hamiltonian (2.4) for investigation of its spectrum. The algebraic scheme we shall discuss is an analogue of the Fock space construction. It is based on the introduction of the 'ground' vectors in the Hilbert space of the system (2.4). These vectors have special properties with respect to the action of the generators of the intrinsic algebra.

First of all, let us note that in the commutative limit $\theta \to 0$ the algebra (6.3) is converted into the intrinsic algebra of the system (2.4) living on the (pseudo)sphere [11]. Indeed, with the rescaling

$$G \to \frac{G}{4c_1 f} \qquad \mathcal{D} \to \frac{\mathcal{D}}{2f} \qquad \bar{\mathcal{D}} \to \frac{\bar{\mathcal{D}}}{2f} \qquad c_1 \to 2\beta$$

where the parameter β is related to the radius of the (pseudo)sphere, the algebra (6.3) changes into the intrinsic algebra which appeared in [11] in the limit $\theta \to 0$.

Since the commutative case is much simpler than the noncommutative one, we shall start the discussion of the covariant algebraic construction from it.

We introduce a vector ('ground' state) of the Hilbert space which obeys the conditions

$$Z\Psi^{(0)} = \mathcal{D}\alpha(G)\Psi^{(0)} \qquad \bar{Z}\Psi^{(0)} = \bar{\mathcal{D}}\beta(G)\Psi^{(0)} \qquad S\Psi^{(0)} = s\Psi^{(0)}. \quad (6.8)$$

Here $\alpha(.)$ and $\beta(.)$ are some functions, which will be defined later, and $s \in \mathbb{R}$. We start with the real eigenvalues of the operator S since no quantization condition follows from the intrinsic algebra. Besides, we know [11] that the eigenvalues are real in general, since the magnetic flux gives a contribution to them and for a noncompact surface it is not quantized.

From the commutation relations for the generating elements Z and $\bar{Z}$ the following constraint on the functions $\alpha(G)$ and $\beta(G)$ appears,

$$\frac{1}{2}\left(G^2 - \frac{c_1^2}{4}\right)\varrho'(G) + G\varrho(G) = G - 4\beta^2 s \quad (6.9)$$

where $\varrho(G) = \alpha(G) + \beta(G)$. The solution of the differential equation is

$$\varrho(G) = 1 - \frac{8\beta^2 s}{G + \frac{c_1}{2}} - \frac{4\beta^2 c_1 m}{G^2 - \frac{c_1^2}{4}}. \quad (6.10)$$

The parameter m is the integration constant. One should regard this constant as an integer since it corresponds to the quantum number of the planar angular momentum of the system [11].

We will look for the eigenstates of the Hamiltonian in the form

$$\Psi = \sigma(G)\Psi^{(0)} \tag{6.11}$$

where we suppose that $\sigma(G)$ is a polynomial. Then the Hamiltonian (2.4) is reduced to the second-order differential operator

$$H'_\lambda = -4\beta^2\left(G^2 - \frac{c_1^2}{4}\right)\frac{\mathrm{d}^2}{\mathrm{d}G^2} + \left(\left(G^2 - \frac{c_1^2}{4}\right)(\alpha(G) - \beta(G)) - 8\beta^2 G\right)\frac{\mathrm{d}}{\mathrm{d}G}$$
$$+ G(2f + \lambda - 1) + \left(G^2 - \frac{c_1^2}{4}\right)\left(\alpha'(G) + \frac{\alpha(G)\beta(G)}{4\beta^2}\right) + \text{const} \tag{6.12}$$

acting on the function $\sigma(G)$.

Let us fix the residuary arbitrariness in the definition of the functions $\alpha(G)$ and $\beta(G)$ requiring that the operator (6.12) is quasi-exactly solvable. Therefore, it should be represented in terms of the generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra [12]

$$J_+ = G^2\frac{\mathrm{d}}{\mathrm{d}G} - (\mathsf{N}-1)G \qquad J_0 = G\frac{\mathrm{d}}{\mathrm{d}G} - \frac{\mathsf{N}-1}{2} \qquad J_- = \frac{\mathrm{d}}{\mathrm{d}G} \tag{6.13}$$

where if the real parameter N is a non-negative integer, it defines dimensionality of the corresponding finite-dimensional representation. Moreover, the operator (6.12) is a quadratic polynomial in the generators (6.13) since it is a second-order differential operator [12].

Eventually, one can conclude that the operator (6.12) can be represented in the form

$$T = c_1^2\beta^2 J_-^2 - 4\beta^2 J_0^2 + a_+ J_+ + a_0 J_0 + a_- J_- + \text{const} \tag{6.14}$$

where the parameters $a_\pm$, a_0 and N have the following eight solutions,

$$\begin{aligned}
&\mathsf{N} = \epsilon\lambda \qquad a_+ = \epsilon \qquad a_0 = 4\epsilon\beta^2(2s-\lambda) \qquad a_- = \frac{\epsilon}{4}c_1(c_1 + 16\beta^2(m-s))\\
&\mathsf{N} = \epsilon(\lambda - m) \qquad a_+ = \epsilon \qquad a_0 = 4\epsilon\beta^2(2s-m-\lambda) \qquad a_- = \frac{\epsilon}{4}c_1(c_1 - 16\beta^2 s)\\
&\mathsf{N} = \epsilon(\lambda - 2s) \qquad a_+ = \epsilon \qquad a_0 = -4\epsilon\lambda\beta^2 \qquad a_- = \frac{\epsilon}{4}c_1(c_1 - 16\beta^2(m-s))\\
&\mathsf{N} = \epsilon(\lambda + m - 2s) \qquad a_+ = \epsilon \qquad a_0 = 4\epsilon(m-\lambda)\beta^2 \qquad a_- = \frac{\epsilon}{4}c_1(c_1 + 16\beta^2 s)
\end{aligned} \tag{6.15}$$

where $\epsilon = \pm 1$. So, if the parameters of the system take such values that $\mathsf{N} \in \mathbb{Z}_+$ for any of the solutions, then the system is quasi-exactly solvable. Any QES part of the spectrum is a superstructure over the corresponding 'ground' state. Therefore, one can interpret these states as some QES 'vacua' of the system. Here and in what follows we will understand by the QES 'vacuum' a state over which one can construct a certain number of states of the QES part of the spectrum following an oscillator-like procedure based on an intrinsic algebra.

The number of $\mathfrak{sl}(2,\mathbb{R})$ representations (6.14) coincides with the number of solutions found in [11]. At the same time, it is worth noting that the method based on the intrinsic algebra is completely covariant since all the generators of the algebra are covariant quantities.

We would like also to recall that this commutative system possesses a discrete symmetry $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, which intertwines the solutions (6.15) between themselves. A detailed discussion of the symmetry can be found in [11].

6.3. The noncommutative case

To investigate the spectral problem for the noncommutative system with the Hamiltonian (2.4) in terms of the intrinsic algebra (6.3), we introduce a 'ground' state obeying the conditions (6.8) but with the only change

$$S\Psi^{(0)} = m\Psi^{(0)} \qquad m \in \mathbb{Z}. \tag{6.16}$$

One can treat this number as an 'azimuthal' quantum number since it corresponds to the existence of 'axial' symmetry in the system with operator S as its generator. We assume this in accordance with the representation (5.1) of the eigenstates. Therefore, in the noncommutative case the conserved quantity S is an analogue of the angular momentum operator. We would like to note that although we use formally the same conditions (6.8), one should remember that the operators $\mathcal{D}$, $\bar{\mathcal{D}}$ and G do not commute and the ordering of the operators is important.

From the commutation relations of the generating elements (6.3) and the definition of the 'ground' state (6.8) it follows that the functions $\alpha(G)$ and $\beta(G)$ are not independent but obey the constraint

$$\begin{aligned} 2c_1 G((1-\theta f)(\alpha(G)+\beta(G)) + \theta\alpha(G)\beta(G)) + \bar{\mathcal{D}}\mathcal{D}(1-\theta f+\theta\alpha(G))\Delta_+\beta(G) \\ + \mathcal{D}\bar{\mathcal{D}}(1-\theta f+\theta\beta(G))\Delta_-\alpha(G) = 2c_1(m + f(2-\theta f)G). \end{aligned} \tag{6.17}$$

This is the noncommutative analogue of the restriction (6.9).

For investigation of the spectral problem it is convenient to represent the Hamiltonian as the linear in Z, $\bar{Z}$ form:

$$\begin{aligned} H_\lambda = -2\theta^{-1}(1-\theta f)(\mathcal{D}\bar{Z} + \bar{\mathcal{D}}Z) + 2c_1\theta^{-1} f(2-\theta f)G(G-\lambda\theta) \\ + 4c_1\theta^{-1}GS + 2c_1 S(S-\lambda) - \theta^{-2}C_1(1-\theta f)^2 + \theta^{-2}C_2. \end{aligned} \tag{6.18}$$

Using the definition of the 'ground' state (6.8), the spectral problem for the Hamiltonian in this form is reduced to that of the finite-difference operator

$$\begin{aligned} \mathsf{H}_\lambda = 2(\theta f - 1)(\mathcal{D}\bar{\mathcal{D}}(1-\theta f+\theta\beta(G))\Delta_+\Delta_- + (\bar{\mathcal{D}}\mathcal{D}\alpha(G) - \mathcal{D}\bar{\mathcal{D}}\beta(G) \\ + 2c_1(1-\theta f)G)\Delta_+ + \theta^{-1}(\bar{\mathcal{D}}\mathcal{D}\alpha(G) + \mathcal{D}\bar{\mathcal{D}}\beta(G))) \\ + 2c_1 f\theta^{-1}(2-\theta f)G(G-\lambda\theta) + 4c_1 m\theta^{-1}G + \text{const} \end{aligned} \tag{6.19}$$

acting on the function $\sigma(G)$. Here the operators $\Delta_\pm$ are defined as the difference derivatives (3.3) but with N changed for G. The operator (6.19) is a finite-difference operator in the variable G since $\bar{\mathcal{D}}\mathcal{D}$ and $\mathcal{D}\bar{\mathcal{D}}$ are the quadratic polynomials in G,

$$\bar{\mathcal{D}}\mathcal{D} = c_1\left(G + \frac{\theta}{2}\right)^2 + \frac{\theta^2 c_0}{4c_1} \qquad \mathcal{D}\bar{\mathcal{D}} = c_1\left(G - \frac{\theta}{2}\right)^2 + \frac{\theta^2 c_0}{4c_1}$$

as this follows from the intrinsic algebra (6.3) and the form of the Casimir operators (6.5), (6.6). The evident advantage of the representation (6.18) is that the finite-difference operator (6.19) is linear in the functions $\alpha(G)$ and $\beta(G)$.

Representing the functions $\alpha(G)$ and $\beta(G)$ in the form

$$\alpha(G) = \frac{\tilde{\alpha}(G)}{\bar{\mathcal{D}}\mathcal{D}} \qquad \beta(G) = \frac{\tilde{\beta}(G)}{\mathcal{D}\bar{\mathcal{D}}}$$

one can see that if the functions $\tilde{\alpha}(G)$ and $\tilde{\beta}(G)$ are polynomials, then the operator (6.19) is a finite-difference operator with polynomial coefficient functions and vice versa.

Let us fix the form of the functions $\tilde{\alpha}(G)$ and $\tilde{\beta}(G)$ by demanding the Hamiltonian (6.18) be a QES operator of the form (A.10) for some values of the parameters. Moreover, we require that the functions obey the constraint (6.18). In comparison with the commutative case the noncommutative one is drastically different. There are four solutions for the functions

$\tilde{\alpha}(G)$ and $\tilde{\beta}(G)$ providing the quasi-exact solvability of the Hamiltonian (6.18). The explicit form of the function is not important. The corresponding solutions for the coefficients in the representation (A.10) are

$$a_{1,2} = 0 \qquad b_{+}^{\pm} = \mp\frac{2c_1}{\theta}(1-\theta f)^{1\pm\mu} \qquad b_0^{\pm} = 2c_1(2j+\mu((1\pm 1)m-\lambda))(1-\theta f)^{1\mp\mu}$$
$$b_{-}^{\pm} = \pm\frac{\theta(c_1-\epsilon c_0)}{2c_1}(\epsilon c_0 - (4j+2\mu((1\pm\mu)m-\lambda)+1)c_1)(1-\theta f)^{1\mp\mu} \tag{6.20}$$

where $\epsilon = \pm 1$ and $\mu = \pm 1$ parametrize the set of solutions. The main difference of these solutions from those of the commutative case (6.15) is the absence of restrictions on the parameter $\mathsf{N} = 2j+1$. Therefore, the operator (6.19) is quasi-exactly solvable since we can always choose the parameter N to be natural.

The functions $\tilde{\alpha}(G)$ and $\tilde{\beta}(G)$ corresponding to the solution (6.20) depend on the parameter j. Therefore, one can consider that it labels the 'ground' states as well,

$$\Psi^{(0)} = \Psi^{(0)}_{m,j}.$$

Here we recovered the dependence of the state on the quantum number m (6.16). Hence, unlike the commutative case, the noncommutative system admits the infinite number of the 'ground' states for every value of the quantum number m. These states serve as QES 'vacua' of the model.

Like the commutative case, the noncommutative system has a discrete symmetry. Indeed, evidently the substitution

$$c_0 \to -c_0 \tag{6.21}$$

converts the solutions (6.20) with $\epsilon = 1$ to those with $\epsilon = -1$ and vice versa. This transformation is similar to the transformations of the commutative case since it converts one QES operator (6.19) into another of the same type. Besides, in this case there is another discrete transformation

$$\theta \to -\theta \qquad f \to -f \qquad \lambda \to -\lambda \qquad m \to -m. \tag{6.22}$$

This mapping induces the following change of the parameters (6.20) and the operators (A.5),

$$b_{\alpha}^{\pm}(\mu) \to b_{\alpha}^{\mp}(-\mu) \qquad J_{\alpha}^{\pm} \to J_{\alpha}^{\mp}$$

where $\alpha = 0, \pm$. The transformation of the generators is provoked by the change of the sign of the parameter θ. This follows from the definitions of the forward and backward derivatives (3.3) and of the generators (A.5) themselves. In this way, the transformation (6.22) intertwines the QES operators corresponding to the solutions (6.20) with $\mu = 1$ and $\mu = -1$, respectively. The whole group of the discrete symmetry consists of four elements including the identity mapping. The transformations (6.21) and (6.22) are associated with the generators of a finite generated Abelian group [23]. This discrete group is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ since all its elements are involutive. So, as in the commutative case, this discrete symmetry is a kind of duality.

7. Discussion and outlook

Let us summarize briefly the obtained results and discuss some open problems that deserve further attention.

The Dolan–Grady relations provide a rich algebraic structure, which arises in numerous branches of mathematical physics. These relations originally appeared in the framework of various integrable models [1–5]. Later it was observed that the Dolan–Grady relations are necessary conditions of the anomaly-free quantization [8, 9] of pseudo-classical 1D systems

with nonlinear holomorphic supersymmetry [7]. This observation revealed the intimate connection of the relations with the QES systems. The algebraic origin of the construction allowed us to apply it to the two-dimensional systems [10, 11]. In the present paper we demonstrated that

- the Dolan–Grady relations arise in low-dimensional matrix models.

The matrix models naturally lead to the appearance of the noncommutative geometry [13, 14, 24]. Using this observation as a hint, we applied the algebraic scheme described in [9] to the two-dimensional systems on a noncommutative background subjected to an external $U(1)$ gauge interaction. We started by postulating that the coordinates form the generalized deformed oscillator algebra [18]. The Hamiltonian of the system was constructed in terms of generating elements realized as gauge covariant derivatives on the noncommutative manifold. We showed that the generating elements obey the Dolan–Grady relations only when

- the noncommutative coordinates generate $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$ or the Heisenberg algebras.

This allowed us to identify the resulting noncommutative manifold, correspondingly, with the fuzzy sphere S^2_θ, the noncommutative hyperbolic plane H^2_θ or the noncommutative plane $\mathbb{R}^2_\theta$. The cases of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ were discussed in detail.

We showed that the parameter f defining the intensity of the quasi-magnetic field has the three critical values: $f = 0$, $f = \theta^{-1}$ and $f = 2\theta^{-1}$. The first case is trivial since it corresponds to the absence of the gauge interaction ($B = 0$). The second case is analogous to the situation for the particle on the fuzzy sphere and the noncommutative hyperbolic plane in the presence of the constant magnetic field $B = \theta^{-1}$ [22] (see also [21] for the case of the noncommutative plane). For this special value the systems are degenerate. In the third case, $f = 2\theta^{-1}$, the quasi-magnetic field vanishes while the gauge interaction is still present in the model since the corresponding connection is not trivial, $f \neq 0$ (see equations (4.8), (4.12)). This resembles the famous Aharonov–Bohm effect, but here its origin roots in the noncommutativity of the configuration space. Note that there is no analogue of such a phenomenon in the constant magnetic field case.

By the reduction of the Hamiltonian to one-dimensional finite-difference operators, it was shown that the constructed noncommutative systems are quasi-exactly solvable for some values of the coupling constant. In the commutative limit the systems on the fuzzy sphere and noncommutative hyperbolic plane lead to QES systems on the corresponding Riemann surfaces [11].

The algebraic content of the given model was revealed by the construction of the intrinsic algebra. Using the notion of intrinsic algebra, in [11] we proposed a general algebraic method of constructing the covariant form of the integrals of motion. Developing this idea, here we demonstrated how

- the spectral problem can be treated in terms of the intrinsic algebra.

The advantage of this approach is its complete covariance. For example, one does not need to impose any gauge conditions since the intrinsic algebra operates with covariant quantities. The application of this method allowed us to show that the noncommutative systems under consideration are quasi-exactly solvable for any value of the coupling constant.

In the noncommutative case the reduction of the Hamiltonian and the application of intrinsic algebra to the spectral problem lead to some finite-difference equations. To investigate their quasi-exact solvability,

- a deformation of the $\mathfrak{sl}(2,\mathbb{R})$ scheme for finite-difference QES systems was proposed.

In this approach the $\mathfrak{sl}(2,\mathbb{R})$ algebra is changed for its nonlinear analogue (A.6). We showed that the deformed scheme is not equivalent to the approach based on the usual $\mathfrak{sl}(2,\mathbb{R})$ algebra [25] if one requires finite-difference QES operators to be the polynomials in the generators of the corresponding algebras.

Surprisingly enough, but the system (2.4) in a non-constant magnetic field on the noncommutative plane (the coordinates form the Heisenberg algebra) is more complicated than the systems on the fuzzy sphere and the noncommutative hyperbolic plane investigated in this paper. Therefore, it is of interest to investigate this system and verify if it inherits such a property as the quasi-exact solvability. Furthermore, it would be interesting to apply the observed analogy between the matrix equations and the parafermionic trilinear commutation relations [16] to construct new solutions. Such a consideration will be presented elsewhere [26].

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Appendix A. Deformed $\mathfrak{sl}(2,\mathbb{R})$ scheme

Here we discuss a deformation of the $\mathfrak{sl}(2,\mathbb{R})$ scheme related to one-dimensional finite-difference operators, and compare the approaches based on the deformed algebra and on the usual one.

In [25] Smirnov and Turbiner proposed the approach to finite-difference QES equations based on the following representation of the Heisenberg algebra,

$$a = \Delta_+ \qquad b = x(1-\theta\Delta_-) \tag{A.1}$$

where x is a real variable and $\Delta_\pm$ are the forward and backward difference operators (3.3) with respect to this variable. These operators obey the standard commutation relation, $[a, b] = 1$.

As in the standard approach to 1D differential QES equations, an operator for the corresponding spectral problem is supposed to be a polynomial in the $\mathfrak{sl}(2,\mathbb{R})$ generators

$$J_+ = b^2 a - (\mathsf{N}-1)b \qquad J_0 = ba - \frac{\mathsf{N}-1}{2} \qquad J_- = a. \tag{A.2}$$

When $\mathsf{N} \in \mathbb{N}$, the space of polynomials $\mathcal{P}_{N-1}(x)$ is invariant with respect to the action of these operators. Hence, any polynomial in the generators (A.2) is an invariant operator on this space as well, i.e. it is a QES operator.

We are interested in the spectral problem for a three-point finite-difference equation,

$$A(x)\Psi(x+\theta) + B(x)\Psi(x) + C(x)\Psi(x-\theta) = E\Psi(x). \tag{A.3}$$

It is claimed in [25] that all such QES equations with a finite number of polynomial eigenfunctions are classified via the cubic polynomial element of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ taken in the representation (A.1):

$$\mathsf{T} = A_+\left(J_+ + \theta J_0^2\right) + A_1 J_0^2\,(\theta J_- + 1) + A_2 J_0 J_- + A_3 J_0 + A_4 J_- + \text{const} \tag{A.4}$$

where the coefficients are arbitrary real parameters. Below we shall show that this operator is not the most general among QES operators corresponding to the three-point problem (A.3).

Let us introduce the finite-difference operators which are linear in the finite-difference derivatives,

$$J_+^\pm = x^2\Delta_\pm - 2jx \qquad J_0^\pm = x\Delta_\mp - j \qquad J_-^\pm = \Delta_\mp. \tag{A.5}$$

The half-integer parameter j ($2j \in \mathbb{Z}_+$) is an analogue of 'spin' in the usual $\mathfrak{sl}(2,\mathbb{R})$ scheme [12]. These linearly independent operators are the finite-difference analogues of the $\mathfrak{sl}(2,\mathbb{R})$ generators, which can be reproduced in the commutative limit, $\theta \to 0$. The operators (A.5) have another useful property of the $\mathfrak{sl}(2,\mathbb{R})$ generators. They have the invariant finite-dimensional subspace, which is the space of polynomials $\mathcal{P}_{2j}(x)$, $2j \in \mathbb{Z}_+$, $\dim \mathcal{P}_{2j}(x) = 2j+1 = \mathsf{N}$. This allows us to use the generators (A.5) for the construction of finite-difference QES operators in a manner analogous to the continuous case. The exhaustive investigation of this construction lies out of the scope of the paper and here we will discuss only some aspects of it.

Both sets J_μ^+ and J_μ^-, $\mu = 0, \pm$, form the nonlinear subalgebras of the form

$$\begin{aligned}
&\left[J_-^\pm, J_+^\pm\right] = 2J_0^\pm \pm \theta(2j-1)J_-^\pm \\
&\left[J_0^\pm, J_+^\pm\right] = J_+^\pm \mp \theta\big(2J_+^\pm J_-^\pm + (2j+1)\big(J_0^\pm + j\big)\big) \\
&\left[J_-^\pm, J_0^\pm\right] = J_-^\pm T_\mp
\end{aligned} \tag{A.6}$$

where the operators

$$T_\pm = 1 \pm \theta J_-^\mp \tag{A.7}$$

are the discrete forward and backward translation generators, $T_\pm\psi(x) = \psi(x \pm \theta)$. The intertwining commutation relations between the sets J_μ^+ and J_μ^- are

$$\begin{aligned}
&\left[J_-^\mp, J_+^\pm\right] = \big(2J_0^\mp \pm \theta J_-^\mp\big)T_\pm && \left[J_+^+, J_+^-\right] = \theta\big(2J_+^+J_0^+ - J_+^+ + (2j+1)J_+^-\big) \\
&\left[J_0^\mp, J_+^\pm\right] = \big(J_+^\pm \pm \theta\big(J_0^\mp + j\big)\big)T_\pm && \left[J_0^+, J_0^-\right] = J_0^+ - J_0^- \\
&\left[J_-^\mp, J_0^\pm\right] = J_-^\mp && [J_-^+, J_-^-] = 0.
\end{aligned} \tag{A.8}$$

It is obvious that in the commutative limit, $\theta \to 0$, the nonlinear commutation relations (A.6), (A.8) tend to the usual $\mathfrak{sl}(2,\mathbb{R})$ algebra. Furthermore, for the realization (A.5) of the nonlinear algebra the following quadratic identities hold:

$$\begin{aligned}
&J_0^\pm J_0^\pm - J_+^\mp J_-^\pm - J_0^\pm \pm \theta\big(J_0^\pm + j\big)J_-^\pm - j(j+1) = 0 \\
&\theta\big(J_0^+ + j\big)\big(J_0^- + j - 1\big) + J_+^- - J_+^+ = 0 \\
&J_+^+J_0^+ - J_+^-J_0^- - (j+1)\big(J_+^+ - J_+^-\big) = 0 \\
&J_+^-T_+ - J_+^+ + 2j\theta\big(J_0^- + j\big) = 0 \\
&\big(J_0^- + j\big)T_- - J_0^+ - j = 0 \\
&J_-^-T_- - J_-^+ = 0.
\end{aligned} \tag{A.9}$$

Taking into account these identities and the commutation relations (A.6), (A.8), the most general three-point operator (A.3), quadratic in the generators (A.5), can be written as

$$\begin{aligned}
H_{\mathrm{QES}} = {} & a_1J_+^+J_+^- + a_2J_+^-J_0^- + b_+^+J_+^+ + b_0^+J_0^+ + b_-^+J_-^+ + b_+^-J_+^- \\
& + b_0^-J_0^- + b_-^-J_-^- + \text{const}.
\end{aligned} \tag{A.10}$$

By construction this operator is quasi-exactly solvable.

The generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra in the representation (A.2) can be represented as polynomials in the operators (A.5),

$$J_+ = \big(J_+^- - \theta\big(J_0^+ + j\big)\big)T_- \qquad J_0 = J_0^+ \qquad J_- = J_-^-. \tag{A.11}$$

Hence, the QES scheme based on the $\mathfrak{sl}(2,\mathbb{R})$ generators (A.2) is included into that based on the operators (A.5). On the other hand, the operators (A.5) can also be represented in terms of (A.2) via the two last equalities in (A.11) and

$$\begin{aligned} J_+^+ &= J_+ + \theta(J_0+j)(\theta(J_0+j)J_- + 2J_0 - 1) \qquad & J_-^+ &= J_-(1+\theta J_-)^{-1} \\ J_+^- &= J_+T_+ + \theta(J_0+j) & J_0^- &= J_0 + \theta(J_0+j)J_-. \end{aligned} \tag{A.12}$$

According to these relations the shift operator T_+ (A.7) is linear in the generator J_-. As we see, the correspondence is not of a polynomial form for the operator J_-^+. Therefore, the $\mathfrak{sl}(2,\mathbb{R})$ scheme (A.2) is not equivalent to that based on the operators (A.5) if one restricts QES operators to a polynomial in the generators of the (A.2) and (A.5) forms. One can also note that the 5-parametric (up to overall shift) QES operator (A.4) is not the most general QES operator in the framework of the $\mathfrak{sl}(2,\mathbb{R})$ scheme. Indeed, for the case $b_3 = 0$ the QES operator (A.10) can be represented as a fifth degree polynomial in the generators (A.2) with seven arbitrary real parameters.

In addition, we note that if one imposes the following restrictions on the parameters of the QES operator (A.10)

$$a_1 = 0 \qquad a_2 = 0 \qquad b_+^- = -b_+^+ \tag{A.13}$$

then it becomes exactly solvable.

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